ex. given that $\vec{x}_1(t) = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}$, $\vec{x}_2(t) = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}$, $\vec{x}_3(t) = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$

are solutions to system
$$(x_1)' = 3x_1 - 2x_2$$
$$(x_2)' = -x_1 + 3x_2 - 2x_3$$
$$(x_3)' = -x_2 + 3x_3$$

$\downarrow$ rewrite as $(\vec{x})' = P\vec{x}$

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\vec{x}$$

$e^{3t}e^{5t} = e^{3t+5t} = e^{8t}$

show solns are linearly independent :

determinant

det det det

$$W = \begin{vmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & -e^{3t} & e^{5t} \end{vmatrix} \overset{+}{=} 2e^t \begin{vmatrix} 0 & -2e^{5t} \\ -e^{3t} & e^{5t} \end{vmatrix} \overset{-}{-} 2e^t \begin{vmatrix} 2e^{3t} & 2e^{5t} \\ -e^{3t} & e^{5t} \end{vmatrix} \overset{+}{+} e^t \begin{vmatrix} 2e^{3t} & 2e^{5t} \\ 0 & -2e^{5t} \end{vmatrix}$$

$$= 2e^t(-2e^{8t}) - 2e^t(2e^{8t} + 2e^{8t}) + e^t(-4e^{8t})$$

recall $\det\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$= -4e^{9t} \quad -8e^{9t} \quad -4e^{9t}$$

$$= -16e^{9t} \neq 0$$

conclude: $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are linearly independent sol'ns

For now, can say that a general sol'n of $n \times n$ HG systems $\vec{x}' = P(t)\vec{x}$ will be a LC formatted as:

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n$$

composed from $n$ linearly independent solns: $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$
where $c_1, c_2, \dots, c_n$ are arbitrary constants

## § 5.2 Eigenvalue Method for HG Systems

further explore constructing general sol'ns of HG $1^{st}$ order linear <u>systems</u> with constant coefficients

to find $n$ linearly independent sol'n vectors, suppose they are in the form:

$$\begin{bmatrix} x_1 \\ \end{bmatrix} \quad \begin{bmatrix} v_1 e^{\lambda t} \\ \end{bmatrix} \quad \begin{bmatrix} v_1 \\ \end{bmatrix}$$

suppose they are in the form:

recall: $y = c_1 e^{rx}$ format

$$\vec{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} e^{\lambda t}$$

$$= \vec{v} e^{\lambda t}$$

where $v_1, \ldots, v_n$ are scalar constants

generalize: $x_i = v_i e^{\lambda t}$ where $i = 1, 2, \ldots, n$

then $(x_i)' = \lambda v_i e^{\lambda t}$

system is in the format:

$$\begin{aligned} (x_1)' &= a_{11} x_1 + \ldots + a_{1n} x_n \\ (x_2)' &= a_{21} x_1 + \ldots + a_{2n} x_n \\ &\vdots \\ (x_n)' &= a_{n1} x_1 + \ldots + a_{nn} x_n \end{aligned} \longrightarrow \begin{bmatrix} (x_1)' \\ (x_2)' \\ \vdots \\ (x_n)' \end{bmatrix} = \vec{x}' \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \ddots & \vdots \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = A$$

can rewrite system as: $(\vec{x})' = A\vec{x}$

use trial sol'n of $\vec{x} = \vec{v} e^{\lambda t}$ with $(\vec{x})' = \lambda \vec{v} e^{\lambda t}$ substitute

$$\lambda \vec{v} \cancel{e^{\lambda t}} = A \vec{v} \cancel{e^{\lambda t}}$$

since $e^{\lambda t}$ is never zero, can cancel factors of $e^{\lambda t}$

as long as $\vec{v}$ is non-zero and $\lambda$ is constant s.t. $A\vec{v} = \lambda \vec{v}$ hold $\longrightarrow$ $\lambda \vec{v} = A\vec{v}$

$$A\vec{v} = \lambda \vec{v}$$

$$A\vec{v} - \lambda \vec{v} = \vec{0}$$

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

$\lambda$ is <u>eigenvalue</u> of $n \times n$ matrix $A$

then non-zero $\vec{v}$ is the associated <u>eigenvector</u>

then non-zero $\vec{v}$ is the associated <u>eigenvector</u>

system has a non-trivial sol'n iff $\det(A - \lambda I_n) = 0$

called characteristic eqn of A

ex. find eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ 2x2

$$\det(A - \lambda I_2) = 0$$

$$\det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 \qquad \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$3 - 4\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0$$

$$\lambda_1 = 5 \quad \lambda_2 = -1$$

note: $n = 2 \Rightarrow$ 2 eigenvalues by Fundamental Theorem of Algebra

$\mathbb{R}$ or $\mathbb{C}$

Thm: let $\lambda$ = eigenvalue of constant coeff. matrix A

for system $(\vec{x})' = A\vec{x}$

if eigenvector $\vec{v}$ is associated with $\lambda$

then $\vec{x}(t) = \vec{v}e^{\lambda t}$ is a non-trial sol'n to system

<u>Distinct Real Eigenvalues</u>

ex. find gen. sol'n of system

$$(x_1)' = 4x_1 + 2x_2$$
$$(x_2)' = 3x_1 - x_2$$

$\downarrow$ rewrite as $\vec{x}\,' = A\vec{x}$

$$(\vec{x})' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

char eqn: $\det(A - \lambda I_2) = 0$

$$\ldots \begin{bmatrix} \ldots \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \ldots \\ x_2 \end{bmatrix}$$

char eqn: $\det(A - \lambda I_2) = 0$

$$\det\left(\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)(-1-\lambda) - 6 = 0$$
$$-4 - 3\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$
$$(\lambda - 5)(\lambda + 2) = 0$$
$$\lambda_1 = 5 \quad \lambda_2 = -2$$

when $\lambda_1 = 5$: $(A - 5I_2)\vec{v} = \vec{0}$

$$\begin{bmatrix} 4-5 & 2 \\ 3 & -1-5 \end{bmatrix} \vec{v} = \vec{0}$$

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

write as a system

$\mathrm{I}$ $\quad -v_1 + 2v_2 = 0$

$\mathrm{II}$ $\quad (3v_1 - 6v_2 = 0) \div -3 \Rightarrow -v_1 + 2v_2 = 0$ $\quad \mathrm{I}$ also

$$2v_2 = v_1$$

choose eigenvector $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

when $\lambda_2 = -2$ $\quad (A + 2I_2)\vec{v} = 0$

$$\begin{bmatrix} 4+2 & 2 \\ 3 & -1+2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$6v_1 + 2v_2 = 0$$
$$3v_1 + v_2 = 0 \Rightarrow v_2 = -3v$$

choose $\vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

solutions: $\lambda_1 = 5$ $\qquad \lambda_2 = -2$

$EV_1$: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad EV_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\vec{x}_1(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$ $\quad \vec{x}_2(t) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

$\vec{r} e^{\lambda t}$

confirm linear independence: $W = \left| \vec{x}_1 \;\; \vec{x}_2 \right| = \begin{vmatrix} 2e^{5t} & e^{-2t} \\ e^{5t} & -3e^{-2t} \end{vmatrix}$

$-6e^{3t} - e^{3t} \neq 0$

yes, $\vec{x}_1$, $\vec{x}_2$ <u>are</u> linearly independent

$\therefore$ gen. sol'n: $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$

$\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

OR

$x_1(t) = 2c_1 e^{5t} + c_2 e^{-2t}$

$x_2(t) = c_1 e^{5t} - 3c_2 e^{-2t}$

// end of Exam 2 material

look for add'l practice problems

review on Mon :)

Exam 2 next Wed